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DESIGN OF A LAMINATED ANISOTROPIC CURVILINEAR BEAM OF MINIMAL WEIGHT

V. V. Alekhin

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The problems of synthesis of laminated bodies is a promising line of investigation in the field of structural optimization. These problems have been studied in a number of papers [1–5] concerning the questions of design of heat-shielding panels, multilayer wave filters, and elastic laminated bodies. The composition of a structure and its geometric dimensions are chosen as control parameters in the problems of synthesis of laminated structures. A control parameter that characterizes a laminated-body structure is a stepwise function with a discrete range of values. Therefore, in deducing the necessary optimum conditions and in constructing a numerical algorithm, one should use the methods of the optimum-control theory. The structure and sizes of the laminated structure are determined in the process of optimization, although the number, sizes, and materials of the layers are not known beforehand.

In the present paper, we consider the problem of synthesis of a multilayer curvilinear beam of minimal weight, which is bent under a uniformly distributed load, from a finite set of elastic homogeneous orthotropic and isotropic materials under given constraints on the beam strength and thickness. The necessary optimum conditions are obtained, a computational algorithm is built, and an example of calculation of the optimum beam is given.

1. Formulation of the Problem. Let a set W consist of k homogeneous orthotropic and isotropic materials. It is required to synthesize a laminated curvilinear beam of minimal weight from the given set.

Let r_1 and r_2 be the radii of the inner and outer surfaces of the curvilinear beam (see Fig. 1) which is hinge-supported at the ends and loaded by an external pressure q [6]. We shall use the common center of the circumferences that constrain the beam as the origin of coordinates and the axis of symmetry as the polar r axis. The support reactions form equal angles β with the axis of symmetry. Let us denote the angle between the end cross sections of the beam by 2φ. By symmetry of the problem, we can consider half of the beam. In the case of a plane stress state, the stress-strain state of the multilayer curvilinear beam is described in the polar coordinate system $(r,\,\theta)$ by the boundary-value problem including the equations of equilibrium

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}=0,\qquad \frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\,\sigma_{r\theta}}{r}=0; \tag{1.1}$$

the relations of Hooke's law

$$\varepsilon_r=\frac{\sigma_r}{E_r}-\nu_{\theta r}\frac{\sigma_\theta}{E_\theta},\quad \varepsilon_\theta=\frac{\sigma_\theta}{E_\theta}-\nu_{r\theta}\frac{\sigma_r}{E_r},\quad 2\,\varepsilon_{r\theta}=\frac{\sigma_{r\theta}}{G_{r\theta}},\quad \nu_{\theta r}\,E_r=\nu_{r\theta}\,E_\theta, \tag{1.2}$$

where the strain-tensor components are expressed in terms of the radial $u_r\,(r,\,\theta)$ and circumferential $u_\theta\,(r,\,\theta)$ shears in the form

$$\varepsilon_r=\frac{\partial u_r}{\partial r},\quad \varepsilon_\theta=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad 2\,\varepsilon_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}, \tag{1.3}$$

and the boundary conditions

— at the curvilinear beam sides

$$\sigma_r(r_1,\,\theta)=0,\quad \sigma_{r\theta}(r_1,\,\theta)=0,\quad \sigma_r(r_2,\,\theta)=-q,\quad \sigma_{r\theta}(r_2,\,\theta)=0, \tag{1.4}$$

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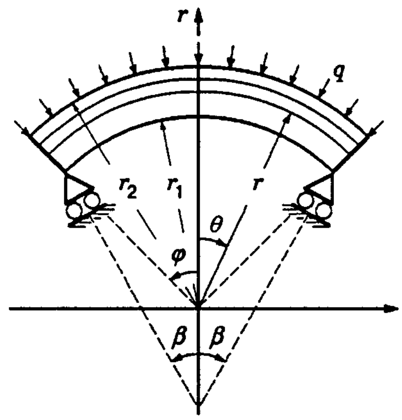

Fig. 1

— at the beam ends, the integral conditions below are satisfied

$$\int_{r_1}^{r_2} \sigma_\theta \, dr = -qr_2 \frac{\sin\varphi \sin(\varphi-\beta)}{\cos\beta}, \quad \int_{r_1}^{r_2} \sigma_{r\theta} dr = qr_2 \frac{\sin\varphi\cos(\varphi-\beta)}{\cos\beta}, \quad \int_{r_1}^{r_2} \sigma_\theta(r-r_1)dr = 0; \tag{1.5}$$

— at the axis of symmetry

$$u_\theta(r, 0) = 0, \quad \sigma_{r\theta}(r, 0) = 0; \tag{1.6}$$

— at the point of the hinge support

$$u_r(r_1, \varphi)\cos(\varphi-\beta) - u_\theta(r_1, \varphi)\sin(\varphi-\beta) = 0. \tag{1.7}$$

Here $E_r(r)$, $E_\theta(r)$, $G_{r\theta}(r)$, $\nu_{\theta r}(r)$, and $\nu_{r\theta}(r)$ are the medium's distributed characteristics: Young's modulus, shear modulus, and Poisson's ratios of the beam-layer materials.

At the inner boundaries $r_i \in (r_1, r_2)$ of the beam layers, where the medium's characteristics undergo a discontinuity, one should specify the conjugation conditions (continuity of shears u_r and u_θ and stresses σ_r and $\sigma_{r\theta}$):

$$[u_r(r_i, \theta)] = [u_\theta(r_i, \theta)] = [\sigma_r(r_i, \theta)] = [\sigma_{r\theta}(r_i, \theta)] = 0. \tag{1.8}$$

Let σ, R, and ρ_* be the quantities having the dimension of stress, length, and density. We introduce new nondimensional variables (below, we omit the asterisk for nondimensional variables):

$$u_i^* = u_i/R, \quad r_i^* = r_i/R, \quad \sigma_{ij}^* = \sigma_{ij}/\sigma, \quad \sigma_{ij}^{\pm *} = \sigma_{ij}^{\pm}/\sigma, \quad E_i^* = E_i/\sigma, \quad G_{r\theta}^* = G_{r\theta}/\sigma, \quad q^* = q/\sigma, \quad \rho^* = \rho/\rho_*$$

($\sigma_{ij}^{\pm}$ and ρ are the strength limits and densities of the materials from the set W). We make a replacement of the coordinates

$$r = r_1 + x(r_2 - r_1), \quad x \in [0, 1], \tag{1.9}$$

which transforms the variable domain of definition of $[r_1, r_2]$ into the constant one $[0, 1]$. We also introduce a stepwise function

$$\alpha(x) = \{\alpha_j;\ x \in [x_j, x_{j+1}),\ j = 1, \ldots, n\}, \quad x_1 = 0, \quad x_{n+1} = 1 \tag{1.10}$$

that characterizes the multilayer-beam structure: the number, sizes, and materials of the layers. The values of α_j belong to the discrete finite set

$$U = \{\alpha_1, \ldots, \alpha_k\}, \tag{1.11}$$

which corresponds to the given set W of materials. Now all characteristics of the materials from the set W are the distribution functions $\alpha(x)$ in the range $[0, 1]$. It is convenient to define a set of integers $U = \{1, \ldots, k\}$

as the set U. In this case, the expression $\alpha(x) = i$, $x \in [x_j, x_{j+1})$, means that the jth beam layer consists of the ith material from the set W.

Since the function $\alpha(x)$ dictates the laminated-beam structure, while the sizes r_1 and r_2 and the aperture angle φ determine its geometry, we shall consider the pair $\{\alpha(x), r_1\}$ as a control parameter (for definiteness, we assume that the outer radius r_2 and the angle φ are fixed). Here $\alpha(x) \in U$ (1.11), and

$$r_1 \in [a, b] \tag{1.12}$$

(a and b are the given limits in which the beam thickness can vary).

The problem of optimum design consists in the following. Among the step functions $\alpha(x)$ (1.10) with the range of values U (1.11) and the parameters r_1 from the region $[a, b]$ it is required to find a pair of control functions $\{\alpha(x), r_1\}$ that minimizes the weight functional

$$F[\alpha, r_1] = 2\varphi \int\limits_{r_1}^{r_2} \rho(\alpha) r\, dr = \int\limits_0^1 \Phi(\alpha, r_1, x) dx \tag{1.13}$$

with the given restriction on the strength

$$\eta(x, \theta, u_r, u_\theta, \sigma_r, \sigma_{r\theta}, \alpha, r_1) \leqslant 0. \tag{1.14}$$

Let us consider the Hoffman strength criterion for unidirectional composites [7] as restriction (1.14). In the case of a plane stress state, this criterion is written in the form

$$\begin{aligned} \eta = \sigma_\theta[(\sigma_\theta - \sigma_r)/(\sigma_\theta^+ \sigma_\theta^-) + (\sigma_\theta^- - \sigma_\theta^+)/(\sigma_\theta^+ \sigma_\theta^-)] + (\sigma_{r\theta}/\sigma_{r\theta}^\pm)^2 \\ + \sigma_r[\sigma_r/(\sigma_r^+ \sigma_r^-) + (\sigma_r^- - \sigma_r^+)/(\sigma_r^+ \sigma_r^-)] - 1 \leqslant 0, \end{aligned} \tag{1.15}$$

where σ_θ^+, σ_θ^-, σ_r^+, σ_r^-, and $\sigma_{r\theta}^\pm$ are the strength limits of materials from the set W under tension and compression in the direction of the θ and r axes and under shear. For isotropic materials, criterion (1.15) transforms to the Mises yield condition. Inequality (1.15) can be written in terms of u_r, u_θ, σ_r, and $\sigma_{r\theta}$ using Hooke's law (1.2).

2. Necessary Optimality Conditions. In order to obtain the necessary optimality conditions in problem (1.1)–(1.15), it is required to construct expressions for variations in goal functional (1.13) and restriction (1.15) in terms of variations in the control pair $\{\alpha(x), r_1\}$. For this purpose, we shall transform boundary-value problem (1.1)–(1.8). The solution of this problem for a homogeneous anisotropic curvilinear beam is given in terms of stresses in [6]. Therefore, in each homogeneous layer of the multilayer beam, the solution of problem (1.1)–(1.8) in terms of shears u_r and u_θ and in terms of stresses σ_r and $\sigma_{r\theta}$ has the form

$$\begin{gathered} u_r(r, \theta) = u_1(r) + u_2(r)\cos\theta + u_3(r)\theta\sin\theta, \\ u_\theta(r, \theta) = v_1(r)\theta + v_2(r)\sin\theta + u_3(r)\theta\cos\theta, \\ \sigma_r(r, \theta) = \sigma_1(r) + \tau(r)\cos\theta, \quad \sigma_{r\theta}(r, \theta) = \tau(r)\sin\theta. \end{gathered} \tag{2.1}$$

Conjugation conditions (1.8) at the inner boundaries of the beam layers and relations (1.9) and (2.1) permit us to introduce the following phase variables which are continuous in the range [0, 1]:

$$\mathbf{Z}(x) = (u_1, v_1, \sigma_1, u_2, v_2, \tau, u_3)^{\mathrm{t}} \tag{2.2}$$

(the superscript t refers to a vector or matrix transposition).

Initial problem (1.1)–(1.8) now can be represented in the form of a boundary-value problem for the unknown $\mathbf{Z}(x)$ (2.2):

$$\mathbf{Z}'(x) = A(\alpha, r_1, x)\mathbf{Z}(x); \tag{2.3}$$

$$z_3(0) = z_5(0) = z_6(0) = 0, \quad z_3(1) = -q, \quad z_6(1) = 0; \tag{2.4}$$

$$\int\limits_0^1 z_6(r_2 - r_1)dx = qr_2 \frac{\cos(\varphi - \beta)}{\cos\beta}, \quad \int\limits_0^1 z_3 r(r_2 - r_1)dx = qr_2\left[r_1 \frac{\sin\varphi\sin(\varphi - \beta)}{\cos\beta} - r_2\right]. \tag{2.5}$$

Here the nonzero elements a_{ij} of the matrix $A(\alpha, r_1, x)$ have the form

$$a_{11} = a_{12} = a_{44} = a_{45} = a_{47} = -\frac{\nu_{\theta r}}{r}(r_2 - r_1), \quad a_{13} = a_{46} = \left(\frac{1}{E_r} - \frac{\nu_{\theta r}^2}{E_\theta}\right)(r_2 - r_1),$$

$$a_{22} = a_{54} = a_{55} = -a_{57} = \frac{r_2 - r_1}{r}, \quad a_{31} = a_{32} = a_{64} = a_{65} = a_{67} = \frac{E_\theta}{r^2}(r_2 - r_1),$$

$$a_{33} = \frac{\nu_{\theta r} - 1}{r}(r_2 - r_1), \quad a_{56} = \frac{r_2 - r_1}{G_{r\theta}}, \quad a_{66} = \frac{\nu_{\theta r} - 2}{r}(r_2 - r_1).$$

Let us clarify how boundary conditions (2.4) and (2.5) are obtained from boundary conditions (1.4)–(1.7). It should be noted that in view of the representation (2.1) and equilibrium equations (1.1), three integral conditions (1.5) are reduced to two integral boundary conditions (2.5), because the first two conditions from (1.5) turn out to be dependent on each other. From an analysis of system (2.3), it then follows that if $z_4(x)$ and $z_5(x)$ are a solution of system (2.3), the functions $\tilde{z}_4(x) = (z_4(x) - d)$ and $\tilde{z}_5(x) = (z_5(x) + d)$, where d is a constant, are also the solutions. Therefore, one can set, for example, $z_5(0) = 0$ in the boundary conditions and find the constant d separately after solving boundary condition (2.3)–(2.5) from boundary condition (1.7), from which it follows that

$$d = \frac{1}{\cos\beta}\{[z_1(0) + z_4(0)\cos\varphi]\cos(\varphi - \beta) - [z_2(0)\varphi + z_5(0)\sin\varphi]\sin(\varphi - \beta) + z_7(0)\varphi\sin\beta\}. \quad (2.6)$$

Boundary condition (1.6) on the axis of symmetry is satisfied automatically by virtue of the representation of the solution in the form of (2.1).

Initial problem (1.1)–(1.8) was thus reduced to solving boundary problem (2.3)–(2.5) for the unknown vector-function $\mathbf{Z}(x)$.

Let us replace the local restriction (1.15) by an equivalent integral restriction

$$F_1[\alpha, r_1, \mathbf{Z}] = 0.5\int_V \{\eta(\ldots) + |\eta(\ldots)|\}dV = \int_0^1 \Phi_1(\alpha, r_1, x, \mathbf{Z}) = 0, \quad (2.7)$$

where V is the volume of the curvilinear beam, while by virtue of the parity of the function $\eta(\ldots)$ with respect to the angle θ in the range $[-\varphi, \varphi]$,

$$\Phi_1(\alpha, r_1, x, \mathbf{Z}) = (r_2 - r_1)[r_1 + x(r_2 - r_1)]\int_0^\varphi \{\eta(\ldots) + |\eta(\ldots)|\}d\theta. \quad (2.8)$$

Note that functional (2.7) has the Frechet derivative [8], because the integrand $|\eta(\ldots)|$, which is a modulus of the Hoffman strength criterion, can vanish in the bent laminated beam only on the zero-measure set consisting of a finite number of points.

Let now the pair $\{\alpha(x), r_1\}$ be the optimum control from the admissible set (1.11) and (1.12) which minimizes the functional (1.13) and satisfies restriction (2.7). Let us consider the perturbed control $\{\alpha^*(x), r_1 + \delta r_1\}$ [8]:

$$\alpha^*(x) = \begin{cases} g(x), & x \in D, \ g(x) \in U, \\ \alpha(x), & x \notin D, \ \text{mes}(D) < \varepsilon, \end{cases} \qquad r_1 + \delta r_1 \in [a, b], \quad |\delta r_1| < \varepsilon \quad (2.9)$$

($D \subset [0, 1]$ is a set of small measure and $\varepsilon > 0$ is a small quantity). Using the standard technique [8], we can obtain the principal parts of the increments of functionals (1.13) and (2.7) [for brevity, the arguments related to the unperturbed control $\{\alpha(x), r_1\}$ are omitted]:

$$\delta F[\ldots] = \int_D \{\Phi(\alpha^*, \ldots) - \Phi(\alpha, \ldots)\}dx + S\delta r_1; \quad (2.10)$$

$$\delta F_1[\ldots] = \int_D \{M(\alpha^*, \ldots) - M(\alpha, \ldots)\}dx + S_1\delta r_1. \quad (2.11)$$

Here

$$M(\alpha, r_1, x, \mathbf{Z}, \mathbf{\Psi}) = \Phi_1(\alpha, r_1, x, \mathbf{Z}) + \mathbf{\Psi}^t(x) A(\alpha, r_1, x)\mathbf{Z}(x) + \gamma_1(r_2 - r_1)z_6(x) + \gamma_2 r(r_2 - r_1)z_3(x), \quad (2.12)$$

$$S = \int_0^1 \frac{\partial}{\partial r_1} \Phi(\alpha, r_1, x)dx, \quad S_1 = \int_0^1 \frac{\partial}{\partial r_1} M(\alpha, r_1, x, \mathbf{Z}, \mathbf{\Psi})dx - \gamma_2 q r_2 \frac{\sin\varphi \sin(\varphi - \beta)}{\cos\beta}.$$

The vector $\mathbf{\Psi}(x)$ of conjugated variables satisfies the boundary-value problem

$$\mathbf{\Psi}'(x) = -A^t(\alpha, r_1, x)\mathbf{\Psi}(x) - \left[\frac{\partial}{\partial \mathbf{Z}} \Phi_1(\alpha, r_1, x, \mathbf{Z})\right]^t - \gamma_1 \mathbf{B} - \gamma_2 \mathbf{C}; \quad (2.13)$$

$$\psi_1(0) = \psi_2(0) = \psi_4(0) = \psi_7(0) = 0, \quad \psi_1(1) = \psi_2(1) = \psi_4(1) = \psi_5(1) = \psi_7(1) = 0, \quad (2.14)$$

where the nonzero components b_i and c_i of vectors $\mathbf{B}$ and $\mathbf{C}$ have the form

$$b_6 = r_2 - r_1, \quad c_3 = [r_1 + x(r_2 - r_1)](r_2 - r_1).$$

The vector $\mathbf{\Psi}(x)$ and the Lagrange multipliers γ_1 and γ_2, which were used for taking into account the integral boundary conditions (2.5) in constructing the variation $\delta F_1[\ldots]$ in restriction (2.7), are determined from boundary-value problem (2.13) and (2.14).

Now let us form an extended functional

$$J[\alpha, r_1] = F[\alpha, r_1] + \lambda_1 F_1[\alpha, r_1, \mathbf{Z}] + \lambda_2\{a - r_1 + \xi_1^2\} + \lambda_3\{r_1 - b + \xi_2^2\} \quad (2.15)$$

(λ_1, λ_2, λ_3 and ξ_1, ξ_2 are the Lagrange multipliers and penalty variables). Let us represent the variation in functional (2.15) using expressions (2.10)–(2.12) as

$$\delta J[\ldots] = \int_D \{H(\alpha, \ldots) - H(\alpha^*, \ldots)\}dx + \{S + \lambda_1 S_1 - \lambda_2 + \lambda_3\}\delta r_1 + 2\lambda_2\xi_1\delta\xi_1 + 2\lambda_3\xi_2\delta\xi_2; \quad (2.16)$$

$$H(\alpha, r_1, x, \mathbf{Z}, \mathbf{\Psi}) = -\Phi(\alpha, r_1, x) - \lambda_1 M(\alpha, r_1, x, \mathbf{Z}, \mathbf{\Psi}). \quad (2.17)$$

Since the control pair $\{\alpha(x), r_1\}$ is optimal (minimizing), the condition $\delta J[\ldots] \geqslant 0$ must be satisfied for any admissible control pair $\{\alpha^*(x), r_1 + \delta r_1\}$ (2.9). Then, in view of the arbitrariness of the variations δr_1 and $\delta\xi_i$, from expression (2.16) we obtain

$$S + \lambda_1 S_1 - \lambda_2 + \lambda_3 = 0; \quad (2.18)$$

$$\lambda_2(a - r_1) = 0, \quad \lambda_3(r_1 - b) = 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0, \quad (2.19)$$

and owing to the fact that the small-measure set D can be dense almost everywhere in the interval [0, 1], the maximum condition for the Hamiltonian $H(\ldots)$ (2.17) with respect to the argument α must be satisfied almost for all $x \in [0, 1]$ [8]:

$$H(\alpha, r_1, x, \mathbf{Z}, \mathbf{\Psi}) = \max_{\alpha^*(x)\in U} H(\alpha^*, r_1, x, \mathbf{Z}, \mathbf{\Psi}). \quad (2.20)$$

Thus, we conclude that the optimum control $\{\alpha(x), r_1\}$ and the corresponding optimum trajectory $\mathbf{Z}(x)$ and the vector $\mathbf{\Psi}(x)$ of conjugate variables must satisfy boundary-value problems (2.3)–(2.5), (2.13), and (2.14), relations (1.11), (1.12), (2.7), and (2.19), and optimality conditions (2.18) and (2.20).

3. Computational Algorithm. The basic idea of the direct method for solving the problems of optimum design consists in constructing a sequence of controls $\{\alpha(x), r_1\}_j$ ($j = 1, 2, \ldots$) that minimizes the goal functional (1.13). For this purpose, introducing a uniform grid $\{x_i\}$, we shall divide the interval [0, 1] into n intervals D_i simulating the sets of small measure. Let us define the initial control $\{\alpha(x), r_1\}$ from the admissible domain (1.11), (1.12), and (2.7). Evidently, $\alpha(x)$ is a stepwise function with constant intervals $D_i = [x_i, x_{i+1})$, in which it takes on the values from the set U (1.11). On a certain set D_i, the next approximation $\{\alpha^*(x), r_1 + \delta r_1\}$ is sought in the form of (2.9)

$$\alpha^*(x) = \begin{cases} \alpha_j, & x \in D, \quad \alpha_j \in U, \\ \alpha(x), & x \notin D; \end{cases} \quad (3.1)$$

TABLE 1

Material	ρ	E_θ	E_r	$G_{r\theta}$	$\nu_{\theta r}$	σ_θ^+	σ_θ^-	σ_r^+	σ_r^-	$\sigma_{r\theta}^\pm$
Fiber-glass plastic	2.13	6070	2488	1197	0.23	129	100	4.6	13	4.6
Carbon-filled plastic	1.61	18140	1035	686	0.28	149.4	110	4	18.6	6.76
Boron plastic	2.02	20130	2172	538	0.17	137.3	120	5.6	20	6.3
Organic plastic	1.36	8430	484	284	0.32	118.6	30	1.1	12	2.76
Sphere plastic	0.65	270	270	106.3	0.27	4.5	4.5	4.5	4.5	2.6
Duralumin	2.85	7100	7100	2669.2	0.33	44	44	44	44	25.4
Titanium alloy	4.6	12,000	12,000	4545.5	0.32	80	80	80	80	46.19
Steel	7.8	21,000	21,000	8076.9	0.3	120	120	120	120	69.28
Copper	8.93	11200	11200	4210.5	0.33	20	20	20	20	11.55

$$r_1 + \delta r_1 \in [a,\, b], \quad |\delta r_1| < \varepsilon \tag{3.2}$$

and is determined from the linearized optimization problem: to find on a given set an admissible perturbation $\{\alpha_j,\, \delta r_1\}$ that ensures a maximum decrease in the functional $F[\ldots]$ (1.13) or, in other words, a minimum of variation $\delta F[\ldots]$ (2.10) under conditions (3.1) and (3.2) and linearized restriction (2.7)

$$F_1[\alpha^*,\, r_1 + \delta r_1,\, \mathbf{Z} + \delta\mathbf{Z}] \approx F_1[\alpha,\, r_1,\, \mathbf{Z}] + \delta F_1[\alpha,\, r_1,\, \mathbf{Z}] = 0, \tag{3.3}$$

where the expression for $\delta F_1[\ldots]$ is given by formula (2.11). This linearized problem is a variant of the problem considered in Secs. 1 and 2. From here we deduce immediately that the optimum perturbation $\{\alpha_j,\, \delta r_1\}$ must satisfy the relations

$$\delta r_1 = -\gamma\{S + \lambda_1 S_1 - \lambda_2 + \lambda_3\}, \quad \gamma \geqslant 0; \tag{3.4}$$

$$\lambda_2(a - r_1 - \delta r_1) = 0, \quad \lambda_3(r_1 + \delta r_1 - b) = 0, \quad \lambda_2 \geqslant 0, \quad \lambda_3 \geqslant 0 \tag{3.5}$$

and restrictions (3.2) and (3.3).

The Lagrange multipliers γ, λ_2, and λ_3 are found from (3.2) and (3.5) in the process of numerical calculation. The best correction α_j (3.1) is determined in the following way. From relations (3.3) and (3.4), we obtain

$$\delta r_1 = -\Big\{ \int\limits_{D_i} [M(\alpha_j, \ldots) - M(\alpha, \ldots)]dx + F_1[\alpha, r_1, \mathbf{Z}] \Big\} / S_1. \tag{3.6}$$

A correction α_j that minimizes the variation $\delta F[\ldots]$ (2.10) is then found from the condition

$$\int\limits_{D_i} H(\alpha_j, r_1, x, \mathbf{Z}, \mathbf{\Psi})dx = \max_{\alpha_* \in U} \int\limits_{D_i} H(\alpha_*, r_1, x, \mathbf{Z}, \mathbf{\Psi})dx$$

$$\Big[H(\alpha_*, r_1, x, \mathbf{Z}, \mathbf{\Psi}) = -\Phi(\alpha_*, r_1, x) + (S/S_1)\, M(\alpha_*, r_1, x, \mathbf{Z}, \mathbf{\Psi})\Big].$$

For $S_1 = 0$, the best correction $\{\alpha_j,\, \delta r_1\}$ is determined from the relations

$$\delta r_1 = -\gamma\{S - \lambda_2 + \lambda_3\}, \quad \int\limits_{D_i} \Phi(\alpha_j, r_1, x)dx = \min_{\alpha_* \in U} \int\limits_{D_i} \Phi(\alpha_*, r_1, x)dx$$

with allowance for restrictions (3.2), (3.3), and (3.5).

Having thus constructed the new control pair $\{\alpha^*(x),\, r_1 + \delta r_1\}$, we assume it as the initial one and construct the next approximation. The process is considered completed on a given division grid $\{x_i\}$ if the control $\{\alpha(x),\, r_1\}$ does not change on any of the sets D_i. The solution obtained is a local minimum in the problem considered.

Example. The set W consists of nine materials having the dimensionless mechanical and physical characteristics presented in Table 1 (we used some data from [9]).

The inner beam surface, whose radius r_1 can vary in the interval [0.75, 0.9], is free from load. The uniformly distributed load $q = 2$ is defined on the outer beam surface whose radius r_2 is considered fixed and equal to unity. The beam aperture angle is $\varphi = 45°$, and the hinge-support angle is $\beta = 30°$. The beam is divided (in thickness) into 50 equal parts which simulate the sets D_i.

A four-layer beam with layers [0.75, 0.765] and [0.78, 0.795] of titanium alloy, [0.765, 0.78] of steel, and [0.795, 1] of duralumin was taken as an initial approximation. As a result of optimization, we obtained a seven-layer beam with inner radius $r_1 = 0.80432$, weight $F^* = 1.4258$, and layers [0.80432, 0.82389] and [0.96086, 0.99609] of carbon-filled plastic, [0.82389, 0.8278] of steel, [0.8278, 0.84345] and [0.84737, 0.96086] of duralumin, [0.84345, 0.84737] of titanium alloy, and [0.99609, 1] of sphere plastic. The titanium-alloy beam with inner radius $r_1 = 0.78831$ and weight $F_* = 2.7354$ is the lightest homogeneous beam satisfying the constraints on the strength (1.15) and body thickness (1.12) under the given load q. For the optimum beam, the relative gain in weight in comparison with the given homogeneous beam is $(1 - F^*/F_*) \cdot 100\% = 47.9\%$.

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